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GRAPHS WITH $3n-6$ EDGES NOT CONTAINING A SUBDIVISION OF K_5

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We determine all graphs on $n \geq 3$ vertices with $3n-6$ edges which do not contain a subdivision of K_5. These are exactly the graphs which one gets from any number of disjoint maximal planar graphs by successively pasting along triangles.

1. Introduction

It was proved in [10] that every (finite, simple) graph G of order $|G| \geq 3$ with edge number $||G|| > 3|G|-6$ contains a subdivision $\dot{K}_5$ of the complete graph K_5. In this paper, we will characterize now the *extremal graphs* of this problem, i.e. the graphs G with $||G|| = 3|G|-6$ not containing a $\dot{K}_5$.

Every triangulation G of the plane, i.e. every maximal planar graph G with $|G| \geq 3$, is an extremal graph. Let us start from vertex disjoint triangulations of the plane $G_1, \dots, G_m$ and choose a triangle $D_i \subseteq G_i$ for $i \in \mathbb{N}_m$. We identify $D_1 \subseteq G_1$ with $D_2 \subseteq G_2$, so getting a graph $G_{1,2}$ from G_1 and G_2. Then one checks $||G_{1,2}|| = 3|G_{1,2}|-6$ and $G_{1,2}$ does, obviously, not contain a $\dot{K}_5$. Now we choose a triangle $D_{1,2} \subseteq G_{1,2}$ and identify $D_{1,2} \subseteq G_{1,2}$ with $D_3 \subseteq G_3$, so getting from $G_{1,2}$ and G_3 a graph $G_{1,2,3}$ with $||G_{1,2,3}|| = 3|G_{1,2,3}|-6$ not containing a $\dot{K}_5$. The triangles which we identify are called *triangles of attachment*. In this way we continue till G_m and get as result an extremal graph. The class of all graphs which we get in this way starting from any sequence $G_1, \dots, G_m$ of vertex disjoint, maximal planar graphs of order at least 4 is denoted by $\mathcal{A}$. As we have seen, all graphs in $\mathcal{A}$ are extremal

Mathematics Subject Classification (2000): 05C83, 05C35 graphs, and the aim of this paper is to prove that every extremal graph G with $|G|\geq 4$ belongs to $\mathcal{A}$.

Theorem 1.1. *If G is a graph of order at least 3 with $||G||=3|G|-6$ not containing a subdivision of K_5, then G arises from vertex disjoint triangulations of the plane by pasting successively along triangles.*

The main tools in the proof of this theorem are the main results from [10] and a result from [8], where K_4^- denotes a K_4 minus one edge.

Theorem M ([10]). *Every 5-connected graph G with $||G||\geq 3|G|-6$ contains a K_4^- or a $\dot{K}_5$.*

Theorem D ([10]). *Every graph G with $||G||>3|G|–6$ and $|G|\geq 3$ contains a $\dot{K}_5$.*

In [8], A. Kézdy and P. McGuinness had deduced some properties of a minor-minimal counterexample to Dirac's conjecture, i.e. the statement of Theorem D. For instance, they had proved that there is no subgraph K_4^- in such a graph. Checking the proof in [8] reveals that they have even shown the following result, where G/e denotes the graph which arises from G by contraction of the edge e.

Theorem KM ([8]). *If the 5-connected graph G does not contain a $\dot{K}_5$, but for every $e\in E(G), G/e$ contains a $\dot{K}_5$, then K_4^- is no subgraph of G.*

(More exactly, the proof yields even the following: If a 5-connected graph G has an edge e contained in at least two triangles such that G/e contains a $\dot{K}_5$, then there is also a $\dot{K}_5$ in G.)

Since a triangle of attachment of a $G\in\mathcal{A}$ is separating the graph G, the 4-connected, maximal planar graphs are the only 4-connected graphs in $\mathcal{A}$. Hence Theorem 1.1 and Theorem D immediately imply a result conjectured by C. Thomassen in [14].

Corollary 1.2. *Every non-planar, 4-connected graph G with $||G||\geq 3|G|–6$ contains a subdivision of K_5.*

We will use the terminology and notation of [10], but we need some supplements. For $\{x,y\}\in\mathcal{P}_2(V(G))$, we define $G\cup[x,y]:=(V(G),E(G)\cup\{[x,y]\})$ and $G–[x,y]:=(V(G),E(G)–\{[x,y]\})$. A subgraph $H\subseteq G$ is *induced*, if $G(V(H))=H$. A graph G is n^+*-connected*, if it is n-connected and every separating vertex set T of G with $|T|=n$ is of the form $T=N_G(x)$ for an appropriate $x\in V(G)$ (this differs somewhat from n^+-connected in [10]). A *cut* T of G is a separating vertex set of G with $|T|=\kappa(G)$. In this paper, the terms T*-fragment* F, *closed* T*-fragment* $\overline{F}:=G(F\cup T)$, and *fragment* F

always refer to a *cut* T of G, and not to any separating set T as in [10]. Then, of course, for a fragment F of G, the cut T belonging to F is $T = N_G(F)$. If H is a closed T-fragment of G, the *corresponding fragment* is $\mathring{H} := H - T$. We speak of a *closed fragment* H *of* G, if $T := N_G(G - V(H))$ is a cut of $G, \overline{H - T}$ is a fragment of G and H is an induced subgraph of G; then $H = \overline{H - T}$ holds. An *end of* G is a minimal element of the ordered set $(\{F : F \text{ fragment of } G\}, \subseteq)$.

For a closed T-fragment H of G, we define $\mathcal{S}_G(H) := \{S \subseteq \mathcal{P}_2(T) - E(G(T))$: *if* $H \cup S$ *contains a subdivision of* K_5, *then also* G *does*$\}$ and set $s_G(H) := \max_{S \in \mathcal{S}_G(H)} |S|$. For instance, if $G(T)$ is not complete, then $\{[x,y]\} \in \mathcal{S}_G(H)$ for all $\{x,y\} \in \mathcal{P}_2(T) - E(G(T))$ and $s_G(H) \geq 1$. For, if $H \cup [x,y]$ contains a subdivision of K_5, we can replace $[x,y]$ with an x,y-path in $G - V(H - \{x,y\})$ and get also a subdivision of K_5 in G.

2. Preliminary results

In this section, we will put together some known general results and properties of graphs in $\mathcal{A}$, which we will use in the proof of Theorem 1.1. We will also use different versions of *Menger's Graph Theorem*, which we will not state, but refer the reader to section 3 in [3] or Chap. IV in [7].

Theorem 2.1 **(Dirac [4]).** *For every set S of n vertices in an n-connected graph G with $n \geq 2$, there is a circuit $C \subseteq G$ containing S.*

For a proof, see, for instance, (2.6)(b) in Chap. XI in [7]. ∎

Theorem 2.2. **(a)** *If T is a minimally separating vertex set in a maximal planar graph G, then $G(T)$ is a circuit.*

(b) (Tutte [15]). *Let G be a 3-connected planar graph. A circuit C of G is a face boundary in some embedding of G in the plane, if and only if it is induced and non-separating. In this case, C is a face boundary in every embedding of G in the plane.*

(a) is proved, for instance, as Theorem 7.2 (v) in [1], for a proof of (b) see, for instance, 4.2.10 in [3] or 2.5.1 in [13]. ∎

We will often use the fact that the "*face boundaries of a 3-connected planar graph*" are independent of the embedding without reference to Theorem 2.2(b). In the next lemma, we will state two known properties of closed fragments, which we will often use in the following. We will leave the easy proof (by Menger's Theorem) to the reader.

Lemma 2.3. **(a)** *Let* $T=\{t_0,t_1,t_2,t_3\}$ *be a cut of a 4-connected graph* G *and let* F *be a* T*-fragment of* G *with* $|F|\geq 2$*. Then there are two disjoint* $\{t_0,t_1\},\{t_2,t_3\}$*-paths in* $G(T\cup V(F))$*.*

(b) *Let* T *with* $|T|=n\geq 2$ *be a cut of the* n*-connected graph* G *and let* F *be a* T*-fragment of* G*. Consider a* $t\in T$*, with* $|N_G(t)\cap F|\geq k\in\mathbb{N}_{n-1}$ *and choose* $T'\subseteq T-\{t\}$ *with* $|T'|+k=n-1$*. Then there is a* $t,(T-(T'\cup\{t\}))$*-fan of order* k *in* $\overline{F}-T'$*.*

We will now put together some properties of the graphs in $\mathcal{A}$. By definition, every $G\in\mathcal{A}$ is constructed from triangulations $G_1,\dots,G_m$ of order at least 4 of the plane pasting successively along triangles. Then $\kappa(G_i)\geq 3$ and if T is a cut of G_i with $|T|=3$, then $G_i(T)$ is a triangle (by Theorem 2.2(a)). If $G_i\not\cong K_4$, but $\kappa(G_i)=3$, therefore, a separating triangle T exists in G_i. Then for the (obviously only) two components C_1,C_2 of G_i-T, the closed T-fragments $\overline{C}_1$ and $\overline{C}_2$ are triangulations of the plane, and pasting $\overline{C}_1$ and $\overline{C}_2$ along T (this means here $\overline{C}_1\cup\overline{C}_2$) gives G_i. Continuing in this way, we see that G can be constructed from graphs $H_1,\dots,H_k$ by pasting along triangles, where every H_i is isomorphic to K_4 or a 4-connected maximal planar graph. These H_i are called the prime factors of G and they are uniquely determined by G, i.e. independent of the decomposition of G into such graphs ([17] or [5]; see also (4.3) in Chap. X in [7] or (1.12) and (1.13) in Chap. 1 in [2]). Then also the triangles of attachment are the same ones in every decomposition of G into prime factors, namely the separating triangles of G. It is also clear that we can begin our procedure of generating $G\in\mathcal{A}$ with any arbitrary prime factor of G. For $G\in\mathcal{A}$, let $\Pi(G)$ and $\mathcal{T}(G)$ denote the set of prime factors and separating triangles of G, respectively.

Prop 1. *For every* $G\in\mathcal{A}$*, the following statments hold.*

(a) *For every edge* $[x,y]$ *of* $P\in\Pi(G),|N_P(x)\cap N_P(y)|=2$ *holds.*

(b) *For an edge* $[x,y]$ *of* $D\in\mathcal{T}(G),|N_G(x)\cap N_G(y)|\geq 3$ *holds.*

(c) *If* $[x,y]\in E(G)$ *is not contained in any* $D\in\mathcal{T}(G)$*, then there is exactly one* $P\in\Pi(G)$ *containing* $[x,y]$*, and we have* $\kappa(x,y;G)=\kappa(x,y;P)$*; in particular,* $\kappa(x,y;G)=3$*, if* $P\cong K_4$*.*

(d) *If* $H\subseteq G$ *is isomorphic to* K_4*, then* $H\in\Pi(G)$ *holds.*

Proof. These properties are easily seen. We remark only for (c), that every induced x,y-path in $G-[x,y]$ is contained in P (cf. (1.5)(i) in Chap. 1 of [2]). Since every complete subgraph of G is contained in a prime factor of G by construction and since a 4-connected triangulation of the plane does not contain a K_4, (d) follows. ∎

Prop 2. *For every* $G\in\mathcal{A}$*, the following statements hold.*

(a) $\kappa(G) \geq 3$.

(b) *If T with $|T| = 3$ separates G, then $G(T) \in \mathcal{T}(G)$.*

(c) *If T with $|T| = 4$ is a cut of $P \in \Pi(G)$, then $P(T) = G(T)$ is a quadrangle.*

(d) *If Q is an induced quandrangle of G, then there is exactly one $P \in \Pi(G)$ with $Q \subseteq P$, and Q separates P, hence also G.*

Proof. (a) is obvious. (b) Since T with $|T| = 3$ does not separate any $P \in \Pi(G)$, it is easy to find $P_1 \neq P_2$ in $\Pi(G)$ with a common $D \in \mathcal{T}(G)$, such that $P_1 - T$ and $P_2 - T$ are in different components of $G - T$. Then $T = V(D)$ holds. (c) is a special case of Theorem 2.2(a). The first claim of (d) follows easily from the construction of G, the second claim follows, since P is a triangulation of the plane and since Q is induced. ∎

The next two properties of graphs in $\mathcal{A}$ are crucial for our proof.

Prop 3. *Let t_0, t_1 and t_0, t_2 with $t_1 \neq t_2$ be pairs of non-adjacent vertices in the graph G. If $G_i := G \cup [t_0, t_i] \in \mathcal{A}$ for $i = 1, 2$, then $[t_0, t_i]$ is in no separating triangle of G_i for $i = 1, 2$, the prime factor P_i of G_i containing $[t_0, t_i]$ is isomorphic to K_4, and $P_1 \cap P_2$ is a triangle t_0, a, b in G. Furthermore, t_1 and t_2 lie in the same component of $G - \{a, b\}$, but this component does not contain t_0.*

Proof. Assume $[t_0, t_1]$ is in $D \in \mathcal{T}(G_1)$. Then $V(D)$ is also separating G_2, but $G_2(V(D))$ is no triangle. This contradicts Prop 2(b).

Hence there is exactly one prime factor P_i of G_i containing $[t_0, t_i]$ for $i = 1, 2$ by Prop 1(c). Let a, b be the common neighbours of $\{t_0, t_1\}$ in P_1 by Prop 1(a). If $|P_1| \geq 5$, then $[a, b] \notin E(P_1)$ by Prop 1(d) and $P_1 - \{t_0, t_1, a, b\}$ is connected. Since $[t_0, t_1]$ is not in a $D \in \mathcal{T}(G_1)$, also $G_1 - \{t_0, t_1, a, b\} = G_2 - \{t_0, t_1, a, b\}$ is connected by construction. But this contradicts Prop 2(d), since $G_2(\{t_0, t_1, a, b\})$ is a quadrangle.

Therefore, $P_i \cong K_4$ for $i = 1, 2$. Since $[t_0, t_1]$ is not in a separating triangle of $G_1, \{a, b\}$ separates t_0 and t_1 in G by Prop 1(c). Since $\{a, b\}$ does not separate t_0 and t_1 in G_2 by Prop 2(a), t_1 and t_2 are in the same component of $G - \{a, b\}$, which does not contain t_0. Since $\{a, b, t_2\}$ separates t_0 and t_1 in G_2, a, b, t_2 is a triangle in G_2 by Prop 2(b), hence also in G. Since $G_2(\{a, b, t_0, t_2\}) \cong K_4, V(P_2) = \{a, b, t_0, t_2\}$ follows by Prop 1(d). ∎

Prop 4. *Let $\{t_0, t_2\}$ and $\{t_1, t_3\}$ be disjoint sets of non-adjacent vertices in the graph G. If $G_i := G \cup [t_i, t_{i+2}] \in \mathcal{A}$ for $i = 0, 1$, then t_0, t_1, t_2, t_3 is a quadrangle in G or there are adjacent vertices x_1, x_2 in G with $N(x_i) \supseteq \{t_0, t_1, t_2, t_3\}$ for $i = 1, 2$ and, furthermore, there are 2 disjoint $\{t_0, t_2\}, \{t_1, t_3\}$-paths in $G - \{x_1, x_2\}$.*

Proof. We distinguish two cases.

(I) $[t_0, t_2]$ *is contained in a* $D \in \mathcal{T}(G_0)$, say, $V(D) = \{t_0, t_2, t\}$.

Then there are at least 2 prime factors P_1 and P_2 of G_0 containing D. By Prop 1(a), there is an $x_i \in P_i - V(D)$ with $N_G(x_i) \supseteq \{t_0, t_2\}$ for $i = 1, 2$, and, obviously, $[x_1, x_2] \notin E(G_0)$. Since t_0, x_1, t_2, x_2 is a quadrangle Q in G, we may assume $\{x_1, x_2\} \neq \{t_1, t_3\}$. Then Q is even an induced quadrangle in G_1 and hence belongs to exactly one prime factor P of $G_1 \in \mathcal{A}$ by Prop 2(d). Obviously, $P \ncong K_4$, hence $\kappa(P) \geq 4$, in particular, $\kappa(x_1, x_2; P) \geq 4$. Since $V(D)$ separates x_1 and x_2 in G_0, hence in G, we may assume that x_1 and t_1 belong to the same component C_1 of $G–V(D)$. Then also x_2 and t_3 are in the same component C_2 of $G - V(D)$, and $C_1 \neq C_2$. If $x_1 \neq t_1$, then $\{t_1\} \cup V(D)$ separates x_1 and x_2 in P, hence is an induced quadrangle by Prop 2(c), hence $N_G(t_1) \supseteq \{t_0, t_2\}$. If $x_1 = t_1$, then $N_G(t_1) \supseteq \{t_0, t_2\}$ by definition of x_1. In the same way, one concludes $N_G(t_3) \supseteq \{t_0, t_2\}$, and so t_0, t_1, t_2, t_3 is a quadrangle in G.

(II) $[t_0, t_2]$ *is not contained in a separating triangle of* G_0.

Then $[t_0, t_2]$ is in exactly one prime factor P of G_0 by Prop 1(c). There are exactly two common neighbours x_1, x_2 of t_0 and t_2 in P by Prop 1(a); set $S := \{t_0, t_2, x_1, x_2\}$. Then one of the following two cases occurs:

(i) $G(S)$ *is a quadrangle* or

(ii) $G_0(S) = P \cong K_4$ (by Prop 1(d)).

Case (i). Hence $|P| \geq 5$ and $P–S$ is connected by Prop 2(c). Since $[t_0, t_2]$ is not contained in a separating triangle of G_0 and since $[x_1, x_2] \notin E(G)$, also $G_0–S = G–S$ is connected and hence $G_1–S$ is connected. But now Prop 2(d) implies that $G_1(S)$ is no quadrangle, hence $\{t_1, t_3\} = \{x_1, x_2\}$, and t_0, t_1, t_2, t_3 is a quandrangle in G.

Case (ii). Then $[x_1, x_2] \in E(G)$ and $\{x_1, x_2\}$ separates t_0 and t_2 in G by Prop 1(c). Since $\kappa(G_1) \geq 3, \{x_1, x_2\}$ cannot separate G_1. Hence $\{x_1, x_2\} \cap \{t_1, t_3\} = \emptyset$ and one element of $\{t_1, t_3\}$, say, t_1, is in the same component of $G - \{x_1, x_2\}$ as t_0, the other element t_3 is in the same component as t_2. So there are a t_0, t_1-path and a t_2, t_3-path in $G - \{x_1, x_2\}$ which are disjoint. Then $\{x_1, x_2, t_i\}$ separates t_0 and t_2 in G_1 for $i = 1, 3$, hence is a triangle in G_1 by Prop 2(b), hence also in G. So the second case in Prop 4 occurs. ∎

3. Proof of Theorem 1.1

In this section, we will show that every graph G of order at least 4 with $||G|| = 3|G| - 6$, but without a $\dot{K}_5$ belongs to $\mathcal{A}$. We assume that this is not true and choose a graph $G \notin \mathcal{A}$ with $||G|| = 3|G| - 6$, but without $\dot{K}_5$ of least order $|G| \geq 4$. Then, obviously, $|G| \geq 5$ and $\delta(G) \geq 3$ holds by Theorem D.

If there is an $x \in G$ with $d(x) = 3$, then $G(N(x))$ must be complete by Theorem D and $G - x \in \mathcal{A}$ by the choice of G. But this would imply $G \in \mathcal{A}$. Hence, we get

$$\delta(G) \geq 4. \tag{1}$$

Since $|G| \geq 5, G$ is not complete. Hence, there is a cut T of G and we have a decomposition of G into closed T-fragments G_1, G_2 with $V(G_1) \cap V(G_2) = T$; define $\mathring{G}_i := G_i - T$ for $i = 1,2$. By (1), $|G_i| \geq 5$ holds for $i = 1,2$. We set $\mathcal{S}_i := \mathcal{S}_G(G_i)$ and $s_i := s_G(G_i)$ for $i = 1,2$. Choose $S_i \in \mathcal{S}_i$ with $|S_i| = s_i$ for $i = 1,2$. By the definition of $\mathcal{S}_i, G_i \cup S_i$ does not contain a $\dot{K}_5$. Therefore, Theorem D implies

$$||G_i|| + s_i \leq 3|G_i| - 6 \text{ for } i = 1,2. \tag{2}$$

By addition, we get

$$3(|G| + |T|) - 12 \geq ||G|| + ||G(T)|| + s_1 + s_2 = 3|G| - 6 + ||G(T)|| + s_1 + s_2,$$

hence,

$$3|T| \geq ||G(T)|| + s_1 + s_2 + 6. \tag{3}$$

If $G(T)$ is not complete, then $s_i \geq 1$ for $i = 1,2$. Hence, (3) implies $|T| \geq 3$, i.e. $\kappa(G) \geq 3$. Let us assume first $|T| = 3$, say, $T = \{t_0, t_1, t_2\}$. If an inaquality in (2) is proper, then also (3) is proper. Therefore, $|T| = 3 = ||G(T)||$ implies $||G_i|| = 3|G_i| - 6$ for $i = 1,2$, hence $G_i \in \mathcal{A}$ for $i = 1,2$ by the minimal choice of G. Since $G(T)$ is a triangle, this would imply $G \in \mathcal{A}$, in contradiction to our assumption. Therefore, $G(T)$ is not complete, hence $s_i > 0$ for $i = 1,2$ and so $||G(T)|| \leq 1$ follow. Hence, there is an isolated vertex t in $G(T)$, say, $t = t_0$. By (1), $d_{G_i}(t_0) \geq 2$ for $i = 1$ or $i = 2$, say, for $i = 1$. Then Lemma 2.3(b) implies $S := \{[t_0, t_i] : i = 1,2\} \in \mathcal{S}_2$, hence $s_2 \geq 2$ and so $||G(T)|| = 0$. The equality in (3) for $s_1 = 1$ and $s_2 = 2$ shows equality in (2) for $G_2 \cup S$, hence $G_2 \cup S \in \mathcal{A}$. Then there are $x_1 \neq x_2$ in $\mathring{G}_2$ with $N(x_i) \supseteq \{t_0, t_i\}$ for $i = 1,2$ by Prop 1(a). But this shows also $s_1 \geq 2$, which is not compatible with (3). So we have shown $|T| \geq 4$, hence

$$\kappa(G) \geq 4. \tag{4}$$

The main difficulty in our proof is to show that G is even 5-connected. This will occupy us for the next pages. So we assume now $|T| = 4$, say, $T = \{t_0, t_1, t_2, t_3\}$. Then (3) implies

$$||G(T)|| + s_1 + s_2 \leq 6. \tag{5}$$

The following property is easy to see.

(6) *There is no triangle in* $G(T)$.

For assume t_1, t_2, t_3 span a triangle. Choose any $z_i \in \mathring{G}_i$ for $i=1,2$. Then by (4), there is a z_i, T-fan $P_0^i, P_1^i, P_2^i, P_3^i$ of order 4 in G_i for $i=1,2$. But then $G(T) \cup \bigcup P_j^i$ contains a $\dot{K}_5$ with branch vertices z_1, z_2, t_1, t_2, t_3.

(6) implies $||G(T)|| \leq 4$ and so $s_i \geq 1$ for $i=1,2$. *First we consider the case* $||G(T)||=4$. Then $G(T)$ is a quadrangle by (6), say, the circuit t_0, t_1, t_2, t_3, and $s_1 = s_2 = 1$ holds by (5). Let us consider $G_1^i := G_1 \cup [t_i, t_{i+2}]$ for $i=0,1$. Equality in (5) implies equality in (2), hence $||G_1^i|| = 3|G_1^i| - 6$ for $i=0,1$. By the choice of G, we have $G_1^i \in \mathcal{A}$ for $i=0,1$. If $[t_i, t_{i+2}]$ is contained in a separating triangle of G_1^i for $i=0,1$, then there is a $z_i \in \mathring{G}_1$ with $N(z_i) \supseteq \{t_i, t_{i+2}\}$ for $i=0,1$ by Prop 1(b). Since $z_1 \neq z_2$ implies $s_2 \geq 2$, we have $z_1 = z_2$. If $|G_1| \geq 6$, then there is a $z \in \mathring{G}_1 - z_1$, and a z, T-fan of order 3 in $G_1 - z_1$, which exists by (4), shows again $s_2 \geq 2$. So only $|G_1| = 5$ remains and t_0, t_1, t_2, t_3, t_0 is the boundary of a face of the planar graph G_1. If, for instance, $[t_0, t_2]$ is not contained in a separating triangle of G_1^0, then, by Prop 1(c), there is a uniquely determined prime factor P of G_1^0 containing $[t_0, t_2]$, hence also t_1, t_3. Then $|P| \geq 5$ and t_0, t_1, t_2, t_3, t_0 is the boundary of a face of $P - [t_0, t_2]$.

In the same way one sees that $G(T)$ is a face quadrangle of G_2 or of $P' - [t_i, t_{i+2}]$ for a prime factor $P' \ncong K_4$ of $G_2^i := G_2 \cup [t_i, t_{i+2}]$ for $i=0$ or $i=1$. But this implies also $G \in \mathcal{A}$. This contradiction shows $||G(T)|| \leq 3$.

The case $||G(T)|| = 3$ can be easily excluded using the following property.

(7) *If* $||G_2|| + 1 = 3|G_2| - 6$, *then* $\delta(G(T)) \geq 2$ *holds.*

Let us assume, for instance, $d_{G(T)}(t_0) \leq 1$, say, $[t_0, t_i] \notin E(G)$ for $i=1,2$. By assumption and the choice of $G, G_2^i := G_2 \cup [t_0, t_i] \in \mathcal{A}$ for $i=1,2$. By Prop 3, there are $H^i \cong K_4^-$ in G_2 containing t_0, t_i for $i=1,2$ which have a common triangle t_0, a, b such that t_1 and t_2 are in the same component of $G_2 - \{a,b\}$, but t_0 is in another one. We have $[t_1, t_2] \notin E(G)$, since, otherwise, $\{t_1, t_2, a, b\}$ would span a K_4 and (4) would provide a $\dot{K}_5$. Hence also $G_2^3 := G_2 \cup [t_1, t_2] \in \mathcal{A}$, and Prop 3 applied to G_2^1 and G_2^3 now gives that for the common neighbours of t_0 and t_1, i.e. a and b, a component of $G_2 - \{a,b\}$ contains both the vertices t_0 and t_2, which is not true as we have seen above.

(8) $\delta(G) \geq 5$.

Suppose there is a $z \in G$ of degree 4. Then $||G'|| + 1 = 3|G'| - 6$ holds for $G' := G - z$ and $\delta(G(N(z))) \geq 2$ holds for the cut $N(z)$ by (7). But this implies $||G(N(z))|| \geq 4$, contradicting the yet proved fact $||G(T')|| \leq 3$ for every cut T'.

Let us assume now $||G(T)||=3$. Then there is a vertex in $G(T)$ of degree at most 1, say, $d_{G(T)}(t_0) \leq 1$. By (8), we may assume $d_{G_2}(t_0) \geq 3$. Then Lemma 2.3(b) implies $s_1 \geq 2$, hence $s_2=1$ by (5). So the preassumption of (7) is satisfied and we get the contradiction $\delta(G(T)) \geq 2$. So we conclude $||G(T)|| \leq 2$.

Let us now consider the case $||G(T)|| = 2$. First assume that $G(T)$ has an isolated vertex, say, t_0. By (8), we may assume $d_{G_2}(t_0) \geq 3$. Then, by Lemma 2.3(b) we have $s_1 \geq 3$, hence $s_2=1$ by (5). Then we can apply again (7) and get the contradiction $\delta(G(T)) \geq 2$. Hence $G(T)$ must consist of two disjoint edges, say, $[t_0,t_1]$ and $[t_2,t_3]$.

Since $|G_i| \geq 6$ by (8), Lemma 2.3(a) provides two disjoint $\{t_0,t_1\},\{t_2,t_3\}$-paths P_i^0, P_i^1 in G_i for $i = 1,2$. This implies $s_1 = s_2 = 2$ by (5). By (8), $d_{G_i}(t_0) \geq 3$ for $i = 1$ or $i = 2$, say, for $i = 1$. Then we get a $t_0,\{t_2,t_3\}$-fan P^2,P^3 of order 2 in $G_1 - t_1$ by Lemma 2.3(b). Say, P_1^j is a t_j,t_{j+2}-path for $j = 0,1$. Then $S := \{[t_0,t_2],[t_1,t_3]\} \in \mathcal{S}_2$ and $G_2' := G_2 \cup S \in \mathcal{A}$. Since $Q := G_2'(T)$ is a quadrangle t_0,t_1,t_3,t_2, by Prop 2(d), there is exactly one $C \in \mathit{\Pi}(G_2')$ containing Q and Q separates C. Hence there is a vertex z_1 in the "interior" and vertex z_2 in the "exterior" of Q in C. Therefore, there is a z_i,T-fan F_i^1,F_i^2,F_i^3,F_i^4 of order 4 in C for $i=1,2$. But then $G(T) \cup P^2 \cup P^3 \cup \bigcup_{i\in\mathbb{N}_2} \bigcup_{j\in\mathbb{N}_4} F_i^j$ contains a $\dot{K}_5$ with branch vertices z_1,z_2,t_0,t_2,t_3. This shows that we have $||G(T)|| \leq 1$.

Now we consider the case $||G(T)|| = 1$, say, $E(G(T)) = \{[t_0,t_1]\}$. From Lemma 2.3(a), we get again $s_i \geq 2$ for $i = 1,2$. Since $d_{G_i}(t_3) \geq 3$ for $i = 1$ or $i = 2$ by (8), say, for $i = 1$, we get $s_2 = 3$ and $s_1 = 2$ by Lemma 2.3(b) and (5). Since $G_2 \cup \{[t_3,t_i] : i = 0,1,2\} \in \mathcal{A}, t_2$ and t_3 have two common neighbours x_1,x_2 in G_2 by Prop 1(a). Hence $d_{G_2}(t_j) \geq 2$ for $j=2,3$ and so $G_1^j := G_1 \cup \{[t_2,t_3],[t_j,t_{j-2}]\} \in \mathcal{A}$ for $j=2,3$. Now we can apply Prop 4 to $G_1' := G_1 \cup [t_2,t_3]$. Since t_0,t_1,t_2,t_3 does not form a quadrangle in G_1', Prop 4 provides the existence of adjacent vertices x_1,x_2 in G_1 with $N_{G_1}(x_i) \supseteq T$ for $i=1,2$. Since there is a $t_2,\{t_0,t_1\}$-fan of order 2 in $G_2 - t_3$ by Lemma 2.3(b), there would be an $\dot{K}_5$ with branch vertices x_1,x_2,t_0,t_1,t_2 in G.

So only the case $||G(T)|| = 0$ *remains*. Let us first assume $d_{G_1}(t_3) = 1$, say, $[t_3,t_3'] \in E(G_1)$. Since $|G_1| \geq 6$ by (8), also $T' := \{t_0,t_1,t_2,t_3'\}$ is a cut of G, hence T' is independent in G. By (8), we may assume $d_{G_1'}(t_0) \geq 3$ for the closed T'-fragment $G_1' := G_1 - t_3$. By Lemma 2.3(b), we have $S_2 := \{[t_0,t_i] : i \in \mathbb{N}_3\} \in \mathcal{S}_G(G_2)$ and by Lemma 2.3(a) we have $S_1 \in \mathcal{S}_G(G_1')$ for some S_1 consisting of two disjoint elements of $\mathcal{P}_2(T')$. By Theorem D, $||G_i''|| \leq 3|G_i''| - 6$ holds for $G_1'' := G_1' \cup S_1$ and $G_2'' := G_2 \cup S_2$. The addition of these inequalities gives $||G||-1+2+3=||G_1''||+||G_2''||\leq 3(|G|+3)-12$, hence the contradiction $||G||\leq 3|G|-7$. This contradiction proves the following assertion.

(9) $$d_{G_i}(t)\geq 2 \textit{ for } i=1,2 \textit{ and all } t\in T.$$

Let us first assume $||G_1||\geq 3|G_1|-8$, hence $||G_1||=3|G_1|-8$ by Theorem D, since $s_1\geq 2$ by Lemma 2.3(a). By (9) and Lemma 2.3(b), we have $S_i:=\{[t_0,t_1],[t_i,t_{i+2}]\}\in\mathcal{S}_1$ for $i=0,1$. Hence $G_1^i:=G_1\cup S_i\in\mathcal{A}$ for $i=0,1$ by the choice of G. Since t_0,t_1,t_2,t_3 is no quadrangle in $G_1':=G_1\cup[t_0,t_1]$, Prop 4 implies the existence of adjacent vertices x_1 and x_2 in G_1 with $N(x_i)\supseteq T$ and a $\{t_0,t_2\},\{t_1,t_3\}$-path in $G_1-\{x_1,x_2\}$, say, a t_2,t_3-path P. Since (9) and Lemma 2.3(b) provide a $t_0,\{t_2,t_3\}$-fan of order 2 in G_2, we have a $\dot{K}_5$ in G with branch vertices x_1,x_2,t_0,t_2,t_3. This contradiction implies (10).

(10) $$||G_i||=3|G_i|-9 \textit{ for } i=1,2.$$

Let us suppose now that there is a vertex $t\in T$ with $d_{G_i}(t)=2$ for $i=1$ or $i=2$, say, $d_{G_2}(t_0)=2$. Hence $d_{G_1}(t_0)\geq 3$ by (8) and $S_2:=\{[t_0,t_i]:i\in\mathbb{N}_3\}\in\mathcal{S}_2$ by Lemma 2.3(b). This implies $G_2':=G_2\cup S_2\in\mathcal{A}$ by (10) and choice of G. Every $[t_0,t_i]\in S_2$ is in G_2' in at least two triangles by Prop 1(a). Since $d_{G_2}(t_0)=2$, say, $N_{G_2}(t_0)=\{a_1,a_2\}$, these triangles must be t_0,t_i,a_1 and t_0,t_i,a_2 for every $i\in\mathbb{N}_3$. Hence $N(a_i)\supseteq T$ holds for $i=1,2$. If $|G_2|\geq 7$, there is a $z\in\mathring{G}_2-\{a_1,a_2\}$ and by (4) there is a z,T-fan F_1,F_2 of order 2 in $G_2-\{a_1,a_2\}$, say, ending in t_1,t_2. Together with a $t_0,\{t_1,t_2\}$-fan of order 2 in G_1-t_3 (by (9) and Lemma 2.3(b)), we get a $\dot{K}_5$ with branch vertices a_1,a_2,t_0,t_1,t_2, if we still join a_1 and a_2 by the path a_1,t_3,a_2.

So we have seen $|G_2|=6$, hence $[a_1,a_2]\in E(G_2)$ by (8). The graph $G_1=G-\{a_1,a_2\}$ is 3-connected, since a cut T' of G_1 with $|T'|\leq 2$ would provide a cut $T'':=T'\cup\{a_1,a_2\}$ of G containing an edge $[a_1,a_2]$ which we had excluded before. Hence there is a circuit C in G_1 with $|C\cap T|\geq 3$ by Theorem 2.1, and this circuit C together with a_1,a_2 gives obviously a $\dot{K}_5\subseteq G$. This contradiction proves (11).

(11) $$d_{G_i}(t)\geq 3 \textit{ for } i=1,2 \textit{ and all } t\in T.$$

Till now our cut T was arbitrary. *Now we assume that $\mathring{G}_2$ is an end of G.* Then G_2 has the following property.

(12) *For every $a\in\mathring{G}_2$, there is a $t_3,\{t_1,t_2\}$-fan of order 2 in $G_2-\{a,t_0\}$.*

Suppose there is a vertex $t\neq t_3$ separating t_3 and $\{t_1,t_2\}$ in $G_2':=G_2-\{a,t_0\}$. Let C be the component of $G_2'-t$ containing t_3. Then $|C|>1$, since $d_{G_2'}(t_3)\geq 2$ by (11) and the independence of T. Hence $N_G(C-t_3)\subseteq\{t_3,t,a,t_0\}=:T'$ and so $C - t_3$ would be a T'-fragment of G properly contained in the end $\mathring{G}_2$. This contradiction proves (12).

Since $G_2 \cup \{[t_0, t_i] : i \in \mathbb{N}_3\} \in \mathcal{A}$ by (10), (11), Lemma 2.3(b), and the choice of G, there is a common neighbour $a \in \mathring{G}_2$ of t_0 and t_1. Then, by (12), we see $S_i := \{[t_0, t_1], [t_i, t_{i+2}], [t_2, t_3]\} \in \mathcal{S}_1$ for $i = 0, 1$ and $S_2 := \{[t_0, t_1], [t_3, t_0], [t_3, t_1]\} \in \mathcal{S}_1$. Therefore, by (10) and the choice of $G, G_1^i := G_1 \cup S_i \in \mathcal{A}$ for $i = 0, 1$ follows. Since t_0, t_1, t_2, t_3 is no quadrangle in $G_1' = G_1 \cup \{[t_0, t_1], [t_2, t_3]\}$, Prop 4 implies the existence of adjacent vertices x_1, x_2 in G_1 with $N(x_i) \supseteq T$ for $i = 1, 2$. Then, obviously, $G_1 \cup S_2$ contains a K_5, hence G contains a $\dot{K}_5$, since $S_2 \in \mathcal{S}_1$. This contradiction proves that there is no separating set of G of order at most 4.

$$(13) \qquad \kappa(G) \geq 5.$$

Now it is not difficult to complete the proof of Theorem 1.1.

(14) *For every $e \in E(G), G/e$ contains a $\dot{K}_5$.*

Consider any $e = [x, y] \in E(G)$. If there are at least three $z_1, z_2, z_3 \in N(x) \cap N(y)$, then there is a circuit $C \subseteq G - \{x, y\}$ containing z_1, z_2, z_3 by Theorem 2.1, since $\kappa(G - \{x, y\}) \geq 3$ by (13). Since this implies $\dot{K}_5 \subseteq G$, we have $|N(x) \cap N(y)| \leq 2$, hence $||G/e|| \geq 3|G/e| - 6$. We may assume $||G/e|| = 3|G/e| - 6$ by Theorem D, hence $|N(x) \cap N(y)| = 2$, say, $N(x) \cap N(y) = \{z_1, z_2\}$, and, furthermore, $G/e \in \mathcal{A}$ by the choice of G. Since $\kappa(G/e) \geq 4$ by (13), G/e is maximal planar. Let z be the vertex of G/e which arises by contraction of e. Then $N_{G/e}(z)$ spans a circuit C in $G/e - z$ and $G' := G/e - (\{z\} \cup V(C))$ is connected by Theorem 2.2(b), since $\kappa(G/e - z) \geq 3$. Let C_1 and C_2 be the two segments of C between z_1 and z_2, and set $\mathring{C}_i := C_i - \{z_1, z_2\}$ for $i = 1, 2$. If we had $\mathring{C}_i \cap N_G(x) = \emptyset$ or $\mathring{C}_i \cap N_G(y) = \emptyset$ for $i = 1, 2$, then G were a maximal planar graph, in particular, $G \in \mathcal{A}$. Hence we may assume $\mathring{C}_1 \cap N(x) \neq \emptyset$ and $\mathring{C}_1 \cap N(y) \neq \emptyset$. Let v be the neighbour of z_1 on C_1, say, $v \in N(x)$, and choose any vertex $u \in \mathring{C}_1 \cap N(y) \neq \emptyset$. Then we get a $\dot{K}_5$ with branch vertices x, y, z_1, z_2, v using the following (non-trivial) paths: the z_1, z_2-path C_2, the v, u-path on C_1 enlarged by the edge $[u, y]$, and a v, z_2-path over G', which exists, since G' is connected and $N_G(c) \cap G' \neq \emptyset$ for all $c \in C$ by (8). This proves (14).

Since G is 5-connected by (13) and does not contain a $\dot{K}_5$, but for every edge $e, G/e$ contains a $\dot{K}_5$ by (14), we know from Theorem KM that G does not contain a subgraph K_4^-. But this contradicts Theorem M, and Theorem 1.1 is proved. ∎

Concluding remarks

Let P be any property for graphs. A graph G is *edge-maximal without* P, if G does not have property P, but for every $\{x,y\} \in \mathcal{P}_2(V(G)) - E(G)$, the graph $G \cup [x,y]$ has property P. Let, for instance, P be the property to contain a $\dot{K}_4$. Then it is well known that the edge-maximal graphs of order at least 3 without P are exactly the graphs which one gets from vertex disjoint triangles by successively pasting along edges (see, for instance, Proposition 8.3.1 in [3]). In [16], all edge-maximal graphs without a minor K_5 have been determined: *one gets them from the maximal planar graphs and a special 3-regular graph* W (which consists of a circuit of length 8 and the 4 main diagonals) *by successively pasting along triangles and* (if not both are in a triangle) *along edges.* (Cf. also Theorem 5 and [6].)

For the remaining, let P be the property to contain a $\dot{K}_5$. Let $\mathcal{B}$ denote *the class of all edge-maximal graphs without a* $\dot{K}_5$. If one hopes that there is a similar construction for $\mathcal{B}$ as in both the cases described above, one should consider the *subclass* $\mathcal{B}_0$ of $\mathcal{B}$ which *consists of all* $G \in \mathcal{B}$ *which have no separating complete graph.* For instance, $\mathcal{A} \subseteq \mathcal{B}$ and every maximal planar graph without a separating triangle is in $\mathcal{B}_0$. Knowing $\mathcal{B}_0$, one gets immediately again all graphs in $\mathcal{B}$ by successively pasting graphs from $\mathcal{B}_0$ along complete graphs. But there is such a lot of graphs in $\mathcal{B}$, that I am afraid, there is no chance to characterize them. In the following, we will give a few examples. In this, $\mathbb{Z}_n$ denotes the integers modulo n.

(1) Define $G_{3m} := (\mathbb{Z}_{3m} \cup \{s_1,s_2,s_3\}, \{[i,i+1] : i \in \mathbb{Z}_{3m}\} \cup \{[s_i,j] : i \in \mathbb{N}_3, j \in \mathbb{Z}_{3m}$, and $j \equiv i (\mod 3)\} \cup \{[s_i,s_j] : i,j \in \mathbb{N}_3$ and $i \neq j\}$. Then one can check $G_{3m} \cup [1,3] \in \mathcal{B}_0$ for all $m \geq 3$.

(2) The graph F from Figure 6 in [11] and the graph H from Figure 1 in [12] are 3-connected and $3 \leq d(x) \leq 4$ holds for all vertices x. One can check that F and H belong to $\mathcal{B}_0$. F has 16 vertices and girth 5, whereas H has 21 vertices and girth 6. *I suppose that* $\mathcal{B}_0$ *contains graphs of arbitrarily large, finite girth.*

(3) Let $G \ncong K_4$ be a 3-regular, 3^+-connected graph. A construction of these graphs is given, for instance, in [18] and [9]. (Notice that for a 3-regular graph non-isomorphic to the (triangular) prism $K_3 \times K_2$, the three concepts "3^+-connected", "$\frac{7}{2}$-edge-connected", and "cyclically 4-connected" coincide.) Let the quadrangle $Q : q_1,q_2,q_3,q_4$ and the graph G' be as described in the following case a or case b.

Case a. Let q_1,q_2,q_3,q_4 be a quadrangle in G and set $G' := G$.

Case b. Let q_1,q_2,q_3,q_4 be an induced path of length 3 in $G \ncong K_3 \times K_2$ and let Q be the quadrangle q_1,q_2,q_3,q_4 in $G' := G \cup [q_1,q_4]$.

It is easy to see that Q is induced in G' and that $G' - V(Q) = G - V(Q)$ is connected. Then one checks without difficulty that the graph $\overline{G} := G' \cup \{[q_1, q_3], [q_2, q_4]\}$ is in $\mathcal{B}_0$. (Notice that $K_{3,3}$ is the only 3-regular , 3^+-connected graph which contains a cut T such that $G - T$ has more than 2 components.)

(4) Let G be a 4-connected graph without a $\dot{K}_5$. We will give some examples below. By addition of some edges to G, one gets a graph $\overline{G} \in \mathcal{B}$. But then even $\overline{G} \in \mathcal{B}_0$ holds, since a separating complete graph of $\overline{G}$ would have at least 4 vertices and hence $\dot{K}_5 \subseteq \overline{G}$ would follow.

(a) Consider m disjoint copies $K^1, \ldots, K^m$ of $K_{3,3}$, where the independent sets of K^i are $\{a_1^i, a_2^i, a_3^i\}$ and $\{b_1^i, b_2^i, b_3^i\}$. With $x_a \neq x_b$ not in $\bigcup_{i \in \mathbb{N}_m} V(K^i)$ define $L_m := \Big(\{x_a, x_b\} \cup \bigcup_{i \in \mathbb{N}_m} V(K^i), \bigcup_{i \in \mathbb{N}_m} E(K^i) \cup \{[b_j^i, a_j^{i+1}] : i \in \mathbb{N}_{m-1}$ and $j \in \mathbb{N}_3\} \cup \{[x_a, a_j^1] : j \in \mathbb{N}_3\} \cup \{[b_j^m, x_b] : j \in \mathbb{N}_3\} \cup \{[x_a, x_b]\}\Big)$. Then $\kappa(L_m) = 4$ holds, and one can check $L_m \in \mathcal{B}_0$ for all $m \in \mathbb{N}$.

(b) The graph $C_m^2 := C_m[\overline{K}_2]$ arises from a circuit C_m of length m replacing every vertex of C_m with two non-adjacent vertices. Then C_m^2 is 4-connected, non-planar and belongs to $\mathcal{B}_0$ for all $m \geq 6$ (see Example (5.2) in [10]).

(c) Define $Z_m := (\mathbb{Z}_m, \{[i, i+j] : i \in \mathbb{Z}_m$ and $j = 1, 2\})$ for $m \geq 6$. Then Z_m is 4-regular and 4-connected, does not contain a $\dot{K}_5$ and for odd m, it is not planar. But, in general, $Z_m \notin \mathcal{B}$, since for x and y of distance at least 3 in Z_m, $Z_m \cup [x, y]$ does also not contain a $\dot{K}_5$ (see Example (5.3) in [10]).

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